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Stresses set up by rotation in limacon-shaped blades

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SUMMARY

This paper deals with finding the stresses in steadily rotating limacon-shaped thin blades of homogeneous isotropic elastic material. The complex potentials, involving real parameters, have been obtained in closed form. Giving suitable values to the parameters involved, the solutions are derived for the following four positions of the axis of rotation: (a) the line of symmetry of the limacon, (b) any line perpendicular to the line of symmetry and lying in the plane of the blade, (c) any line passing through a point on the line of symmetry and perpendicular to the plane of the blade, (d) normal axis passing through the centroid. The hoop stress and its stationary points have been calculated in all the cases. The results have also been derived for circular blades. The numerical results are presented in the form of tables and graphs.

1. Introduction

The problem of finding the stresses in plates rotating steadily about normal axes have been solved by Love [1], Timoshenko and Goodier [2], Mindlin [3] and Sen [4]. Stevenson [5] used the complex-variable method to analyze the stresses set up by steady rotation in an elliptic plate. Solutions for plates in the form of a cardioid and an epitrochoid have been obtained by Mitra [6] and Madan Mohan [7] respectively, by using Muskhelishvili's method.

The problem of a circular and a triangular plate rotating about a diameter and a side respectively have been solved by Hodge [8] by using a proper stress function. Sen Gupta [9] used Sen's [10] method to solve the problem considered by Madan Mohan [7] earlier. He considered a few interesting particular cases. Arkilic [11] applied the same method to solve the problem of curvilinear polygonal plates rotating about an axis lying in the middle plane of the plates.

The author [12] applied the complex-variable method to solve the problem considered by Arkilic [11]. The comparison of author's solution with that obtained by Arkilic revealed a mistake in the latter solution which had been obtained by a different method. Incidentally it was proved that the method applied by Arkilic did not give the unique solution. The author [13] and Dhaliwal and Chowdhury [14, 15] have considered the rotation of a cardioid about the initial line and that of Booth's lemniscate about the lines of symmetry and the normal axis passing through its centre respectively.

All the above investigations, except [3], are limited to rotation about axes passing through the centroid of the plate. In this paper the problem of a limacon-shaped blade rotating steadily about axes which in general do not pass through the centroid of the limacon, has been solved by using the complex-variable method. The problem has been reduced to a statical one by treating the inertia forces as body forces. It has been noticed that the body forces will have a resultant due to the non-balance of inertia forces. The problem of a limacon rotating about an axis through its centroid perpendicular to the plane of the plate has been solved recently by Tilley [16], which is a particular case of the present work. The comparison of our results with those of Tilley reveals a mistake in his results which is pointed out in detail in Section 8. Results for the rotation of a circular blade about a chord or about any line normal to the plane of the blade and passing through any point within or on the boundary of the circle have been derived. It is believed that the problem of rotation of a circular disc about a chord has not been solved previously. The problem of an eccentrically rotating circular disc has been solved by Mindlin [3] by using bipolar coordinates. Comparison of our results of Section 10 with those of Mindlin [3] show the simplicity of our results which makes it possible to obtain the numerical results easily.

2. Basic equations

The stress components may be put in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x + iy$ and the body-force potential $V(z,\bar{z})$ as

$$\widehat{xx} + \widehat{yy} = 2[\phi'(z) + \bar{\phi}'(\bar{z})] + \epsilon(1+\eta)\, V, \tag{1a}$$

$$\widehat{yy} - \widehat{xx} + 2i\widehat{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] - 4\epsilon(1-\eta)\,\frac{\partial^2 V_1}{\partial z^2}\,. \tag{1b}$$

E, η and ϵ are Young's modulus, Poisson's ratio and density respectively, V_1 is the particular integral of $\nabla^2 V_1 = V$ and overbars are used to denote conjugate complex quantities. The boundary condition for the unstressed boundary may be put in the form

$$\phi(z) + z\,\bar{\phi}'(\bar{z}) + \bar{\psi}(\bar{z}) = \gamma\epsilon\,\frac{\partial V_1}{\partial \bar{z}} - \epsilon \int V dz \tag{2}$$

where

$$\gamma = 2(1-\eta). \tag{3}$$

We introduce the mapping function, which maps the region occupied by the blade in z-plane onto the unit circle in the ζ-plane:

$$z = g(\zeta), \qquad \zeta = \rho e^{i\theta} \tag{4}$$

where ρ and θ may be taken as polar coordinates of the point ζ in the ζ-plane and the corresponding point $z = g(\rho e^{i\theta})$ of the z-plane has ρ and θ as its curvilinear coordinates. The stress components with reference to the curvilinear coordinates may be derived from (1); the results are as follows:

$$\widehat{\rho\rho} + \widehat{\theta\theta} = 2[\phi(\zeta) + \overline{\phi}(\overline{\zeta})] + \epsilon(1+\eta)V, \tag{5a}$$

$$\widehat{\theta\theta} - \widehat{\rho\rho} + 2i\widehat{\rho\theta} = 2\,\frac{\zeta^2 g'(\zeta)}{\rho^2\,\overline{g}'(\overline{\zeta})}\left[\frac{\overline{g}(\overline{\zeta})}{g'(\zeta)}\,\phi'(\zeta) + \Psi(\zeta) - 2\epsilon(1-\eta)\,\frac{\partial^2 V_1}{\partial z^2}\right], \tag{5b}$$

where

$$\phi(z) = \phi[g(\zeta)] \equiv \phi(\zeta), \qquad \psi(z) = \psi[g(\zeta)] \equiv \psi(\zeta),$$

$$\Phi(\zeta) = \phi'(\zeta)/g'(\zeta), \qquad \Psi(\zeta) = \psi'(\zeta)/g'(\zeta).$$

The boundary condition (2) is now transformed into

$$\phi(\sigma) + \frac{g(\sigma)}{\overline{g}'(\overline{\sigma})}\,\overline{\phi}'(\overline{\sigma}) + \overline{\psi}(\overline{\sigma}) = F(\sigma) \tag{6}$$

where $\sigma = e^{i\theta}$ is the value of ζ on the boundary of the unit circle and $F(\sigma)$ is the value of the right-hand side of (2) at $\zeta = \sigma$.

Part I. LIMACON-SHAPED BLADES

3. General solution

Let the body-force potential be

$$V = \frac{1}{8}\,\omega^2\,[p(z^2 + \overline{z}^2) - 2qz\overline{z}] \tag{7}$$

where ω is the uniform angular velocity p, q are numerical constants such that for $p = q = 1$; $p = -1$, $q = 1$; $p = 0$, $q = 2$; V corresponds to a rotation about the x-axis, y-axis and an axis passing through the origin of coordinates and normal to the plane of the blade, respectively. The limacon, shown in Fig. 1, bounded by the curve

$$x = c(n + \cos\theta + m\cos 2\theta),$$

$$y = c(\sin\theta + m\sin 2\theta) \tag{8}$$

is mapped onto the interior of the unit circle in the ζ-plane by the transformation

$$z = g(\zeta) = c(n + \zeta + m\zeta^2), \quad |m| \le \frac{1}{2}, \quad |n+m| \le 1 \tag{9}$$

where $c(> 0)$, m and n are real constants. It may be noticed that the y-axis changes its position with n, it passes through A when $n = 1-m$ and through B when $n = -1-m$.

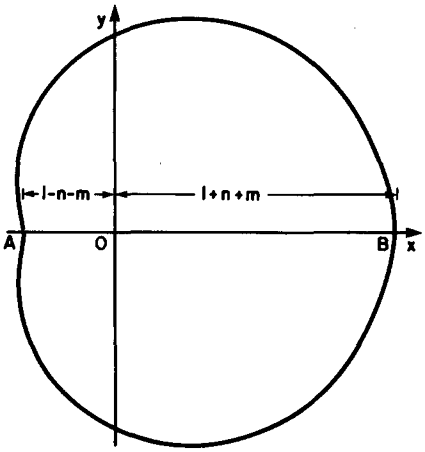

Figure 1. Limacon-shaped blade.

Making use of (4) and (7) in (6), we find that

$$F(\sigma) = K\gamma[p\{g^3(\sigma) + 3g(\sigma)\bar{g}^2\left(\frac{1}{\sigma}\right)\} - 3qg^2(\sigma)\bar{g}\left(\frac{1}{\sigma}\right)]$$

$$- 12K\int [p\{g^2(\sigma) + \bar{g}^2\left(\frac{1}{\sigma}\right)\} - 2qg(\sigma)\bar{g}\left(\frac{1}{\sigma}\right)]g'(\sigma)d\sigma \tag{10}$$

where

$$K = \frac{1}{96}\epsilon\omega^2c^3. \tag{11}$$

Substituting the value of $g(\sigma)$ from (9) in (10) and making some simplifications, we find that

$$F(\sigma) = K\,[L\log\sigma + \sum_{r=-4}^{6} L_r\sigma^r] \tag{12}$$

where

$$L(p,q) = -24[n(1+2m^2) + m](p-q),$$

$$L_6(p,q) = (\gamma-4)m^3p,$$

$$L_5(p,q) = 3(\gamma-4)m^2p,$$

$$L_4(p,q) = 3(\gamma-4)m\,[nm(p-q) + p],$$

$$L_3(p,q) = (\gamma-4)\,[(p-q)(6nm+1) + (1-3m^2)] + 4qm^2,$$

$$L_2(p,q) = 3(\gamma-4)n(1+2nm)(p-q) - 3mq\,[(\gamma-8)(2+m^2) + 4],$$

$$L_1(p,q) = p[6(\gamma-4)n(n+m) - 24nm] - 3q[(\gamma-8)(2n^2 + 2nm + 1 + 2m^2) + 8(n^2-m^2)],$$

$$L_0 = 0,$$

$$L_{-1}(p,q) = 3p[(\gamma+4)(2n^2+2nm+1+2m^2)-8(n^2-m^2)]-3qn[(\gamma+8)(n+2m)-8(n+m)],$$
$$L_{-2}(p,q) = 3p[\gamma(2n^2m+n+2m+m^3)+4m(1+m^2)]-3\gamma qn^2m,$$
$$L_{-3}(p,q) = pm[3\gamma(2n+m)-4m],$$
$$L_{-4}(p,q) = 3\gamma pnm^2. \tag{13}$$

Since the axis of rotation does not pass through the centroid of the blade, unless $p = q$ there will be non-balance of the inertia forces. The inertia forces treated as body forces will have a resultant P acting at the point $x = cn$. Hence the functions $\phi(\zeta)$ and $\psi(\zeta)$ should be holomorphic within the unit circle except at $\zeta = 0$, which is a point of singularity, and may be taken in the form:

$$\phi(\zeta) = \phi_0(\zeta) + \frac{1}{8}(4-\gamma)PK\log\zeta,$$

$$\psi(\zeta) = \psi_0(\zeta) - \frac{1}{8}(4+\gamma)PK\log\zeta \tag{14a}$$

where

$$\phi_0(\zeta) = K\sum_{r=1}^{\infty} a_r\zeta^r, \quad \psi_0(\zeta) = K\sum_{r=1}^{\infty} b_r\zeta^r, \tag{14b}$$

$$P(p,q) = -24[n(1+2m^2)+m](p-q) \tag{14c}$$

and a_r, b_r are real constants.

It may be noticed from (14c) that $P = 0$ when $p = q$ which means that the blade is rotating about the x-axis and the centroid obviously lies on it. Again $P = 0$ when

$$n = \frac{m}{1+2m^2}. \tag{15}$$

For n given by (15), it can be easily verified from (8) that the centroid of the blade will be the origin of the coordinates which shows that the axis of rotation will pass through the centroid and the vanishing of P is justified.

Substituting from (12) and (14) in (6), we get

$$\phi_0(\sigma) + \frac{g(\sigma)}{\bar{g}'(\frac{1}{\sigma})}\left[\phi_0(\frac{1}{\sigma}) + \frac{1}{8}(4-\gamma)P\sigma\right] + \psi_0(\frac{1}{\sigma}) = \sum_{r=-4}^{6} L_r\sigma^r. \tag{16}$$

Multiplying both sides of (16) by $d\sigma/2\pi i(\sigma-\zeta)$ and integrating over the unit circle, we obtain

$$\phi_0(\zeta) + \frac{1}{8}(4-\gamma)P[m\zeta^3+(1-2m^2)\zeta^2+(n-2m+4m^3)\zeta] + ma_1\zeta^2 +$$
$$+[(1-2m^2)a_1+2ma_2]\zeta = \sum_{r=1}^{6} L_r\zeta^r. \tag{17}$$

Comparison of like powers of ζ on the two sides of (17) gives the following values of a_k:

$$a_1(p,q) = \frac{1}{2}(1-2m^2)^{-1}[3(\gamma-4)(p-q)(n^2-6n^2m^2+3nm-16nm^5+4m^2-8m^4)$$

$$-24(p-q)nm-3(\gamma-8)q(1-2m^4)+6\gamma qm^2],$$

$$a_2(p,q) = \frac{1}{2}(1-2m^2)^{-1}[3(\gamma-4)(p-q)\{n^2m(3-2m^2)-nm^2(3-8m^2)$$

$$-2m(1-2m^2)+24(p-q)nm^2-3(\gamma-8)q(3-4m^2-2m^4)m-24qm],$$

$$a_3(p,q) = (\gamma-4)[3(p-q)nm(1-2m^2)+p(1-3m^2)]+4qm^2$$

$$a_r(p,q) = L_r(p,q), \quad r=4,5,6,$$

$$a_r = 0, \quad r \geq 7. \tag{18}$$

Again multiplying the conjugate complex of (16) by $d\phi/2\pi i(\sigma-\zeta)$ and integrating over the unit circle, we obtain

$$\psi_0(\zeta) = -\frac{g(\frac{1}{\zeta})}{g'(\zeta)}\phi_0(\zeta) + ma_1\zeta^{-2} + [(1-2m^2)a_1 + 2ma_2]\zeta^{-1} +$$

$$+L_{-1}\zeta + L_{-2}\zeta^2 + L_{-3}\zeta^3 + L_{-4}\zeta^4. \tag{19}$$

Substituting from (14), (18) and (13) in (19) and simplifying we obtain

$$\psi_0(\zeta) = \frac{K}{1+2m\zeta}\sum_{r=1}^{5} C_r\zeta^r \tag{20a}$$

where

$$C_1(p,q) = 3p[\gamma\{2n^2-nm(1+2m^2)+m^2\}+4\{nm(5-2m^2)+2+5m^2\}]$$

$$-3q[\gamma\{n^2-nm(1-2m^2)\}+4m\{n(5-2m^2)+m\}]-2na_2(p,q),$$

$$C_2(p,q) = 3p[3(\gamma+4)nm\{n(1+2m^2)+m\}+4(n+7m)(1+2m^2)]-$$

$$-18qnm[(\gamma-4)nm^2+2(n+3m)],$$

$$C_3(p,q) = 6(\gamma-8)qn^2m^2+16pm(3n^2m+3n+5m+3m^3),$$

$$C_4(p,q) = 4pm^2(15n+4m),$$

$$C_5(p,q) = 24pnm^3. \tag{20b}$$

The complex functions $\phi(\zeta)$ and $\psi(\zeta)$ are now completely determined by (14), (18) and (20).

4. Rotation about the x-axis

It is easily seen from (7) that for this case $p = q = 1$. Now setting $p = q = 1$ in the results of Section 3, we find that $V(z,\bar{z})$, $\phi(\zeta)$and $\psi(\zeta)$ are given by

$$V = \frac{1}{8}\,\omega^2(z^2+\bar{z}^2-2z\bar{z}), \tag{21}$$

$$\phi(\zeta) = K\sum_{r=1}^{6} P_r\zeta^r,\qquad \psi(\zeta) = \frac{K}{1+2m\zeta}\sum_{r=1}^{5} P'_r\zeta^r \tag{22a}$$

where

$$P'_r = C_r(1,1),\qquad P_r = a_r(1,1). \tag{22b}$$

Inserting the values of $V(z,\bar{z})$, $\phi(\zeta)$ and $\psi(\zeta)$ from (21) and (22) in (5a) and noting that $[\widehat{\rho\rho}]_{\zeta=\sigma} = 0$, we obtain the hoop stress

$$[\widehat{\theta\theta}]_{\zeta=\sigma} = \frac{\epsilon\omega^2c^2}{16(1+4m^2+4m\cos\theta)}\left[\left\{\gamma\left(1+7m^2+4m^4-\frac{m^2}{1-2m^2}\right)+8m^2\,\frac{1-4m^2}{1-2m^2}\right\}\right.$$
$$\left.+\,4\gamma m(1+2m^2)\cos\theta+2\gamma m^2\cos 2\theta\right]. \tag{23}$$

It is seen from (23) that the hoop stress is symmetrical about the x-axis and that the stationary points of the hoop stress occur at $\theta = 0\ (z = n+m+1)$, $\theta = \pi(z = n+m-1)$ and $\theta = \theta_m$ given by

$$\cos\theta_m = \frac{1}{4m}\left[\{(1-4m^2)(1+2m^2+32m^2\gamma^{-1})(1-2m^2)^{-1}\}^{1/2}-(1+4m^2)\right]. \tag{24}$$

It is further noticed that $\theta = \theta_m$ is a point of minimum and $\theta = 0, \pi$ are the points of maxima and that the greatest hoop stress occurs at $\theta = \pi$.

Setting $\theta = \pi$ in (23), we find the greatest hoop stress

$$[\widehat{\theta\theta}]_{\zeta=-1} = \frac{\epsilon\omega^2c^2}{32(1-2m)}\left[\gamma(1-6m+6m^2-4m^3)+\frac{1+2m}{1-2m^2}(\gamma+16m^2)\right]. \tag{25}$$

TABLE 1

Variation of the greatest hoop stress ($\eta = \frac{1}{3}$)

m	0	0.1	0.2	0.25	0.3	0.35	0.4	0.45
$[\widehat{\theta\theta}/\epsilon\omega^2c^2]_{\zeta=-1}$	0.083	0.095	0.157	0.234	0.384	0.697	1.461	5.077

TABLE 2

Variation of the hoop stress $\widehat{\theta\theta}/\epsilon\omega^2c^2$ $(\eta = \frac{1}{3})$

m^{θ}	0°	30°	60°	90°	120°	150°	180°
0	0.083	0.083	0.083	0.083	0.083	0.083	0.083
0.1	0.089	0.089	0.088	0.089	0.090	0.093	0.095
0.2	0.102	0.101	0.099	0.099	0.109	0.134	0.157
0.3	0.116	0.114	0.110	0.110	0.129	0.218	0.384
0.4	0.127	0.124	0.116	0.111	0.134	0.271	0.461

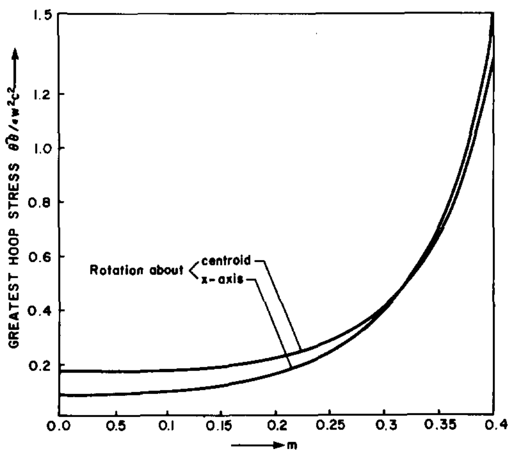

Figure 2. Numerical values of the greatest hoop stress $(\widehat{\theta\theta}/\epsilon\omega^2c^2)_{\zeta=-1}$ against m for a limacon-shaped blade rotating about the x-axis and about a normal axis passing through the centroid.

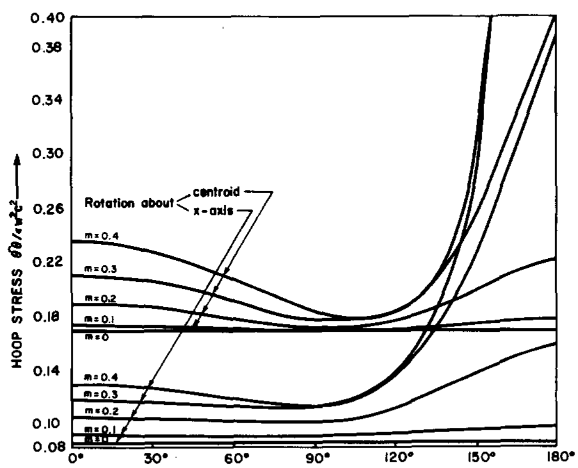

Figure 3. Numerical values of the hoop stress $\widehat{\theta\theta}/\epsilon\omega^2c^2$ against θ $(0° < \theta < 180°)$ for $m = 0.04, 0.2, 0.3, 0.4$ for a limacon-shaped blade rotating about the x-axis and about a normal axis passing through the centroid.

The variation of the greatest hoop stress for different values of $m \leq 0.45$ $(\eta = \frac{1}{3})$ is given in Table 1 and is illustrated graphically in Fig. 2. The variation of the hoop stress from $\theta = 0$ to $\theta = \pi$ for $m = 0, 0.1, 0.2, 0.3, 0.4$, $(\eta = \frac{1}{3})$ is listed in Table 2 and is shown graphically in Fig. 3.

5. Rotation about the y-axis

This case corresponds to $p = -1$, $q = 1$ and from Section 3 we find that for these values of p and q:

$$V = \frac{1}{8}\,\omega^2(z^2 + \bar{z}^2 + 2z\bar{z}), \tag{26}$$

$$\phi(\zeta) = K\Big[\frac{4-\gamma}{8}\,Q\log\zeta + \sum_{r=1}^{6} Q_r\zeta^r\Big],$$

$$\psi(\zeta) = K\Big[-\frac{4+\gamma}{8}\,Q\log\zeta + \frac{1}{1+2m\zeta}\sum_{r=1}^{5} Q_r'\zeta^r\Big] \tag{27a}$$

where

$$Q = P(-1,1), \quad Q_r = a_r(-1,1), \quad Q_r' = C_r(-1,1). \tag{27b}$$

Substituting from (26) and (27) in (5a) and noting that $[\widehat{\rho\rho}]_{\zeta=\sigma} = 0$, we obtain the hoop stress

$$\begin{aligned}[\widehat{\theta\theta}]_{\zeta=\sigma} = {} & \frac{\epsilon\omega^2c^2}{8(1-2m^2)(1+4m^2+4m\cos\theta)}\,[(\gamma-4)\{n^2(1-2m^2-8m^4)+nm(5-4m^2-16m^4) \\ & + \frac{1}{2}(1+12m^2-26m^4-8m^6)\} + 8nm(1-4m^2) + 2(1+6m^2-18m^4-8m^6) \\ & + 2(1-4m^4)\{(\gamma-4)[n(1+2m^2)+2m]+4m\}\cos\theta \\ & + m(1-2m^2)\{(\gamma-4)[2n(1+2m^2)+3m]+4m\}\cos 2\theta].\end{aligned} \tag{28}$$

It is found that the maximum hoop stress occurs at $\theta = 0$ and $\theta = \pi$. The point of minimum hoop stress varies with the values of m and n.

6. Rotation about a normal axis passing through the origin

When the blade is rotating about an axis normal to the plane of the blade and passing through the origin of coordinates, the body-force potential is represented by:

$$V = -\frac{1}{2}\,\omega^2 z\bar{z} \tag{29}$$

which corresponds to $p = 0, q = 2$. For these values of p and q we find from Section 3 that:

$$\phi(\zeta) = K\left[\frac{4-\gamma}{8} R \log \zeta + \sum_{r=1}^{4} R_r \zeta^r\right],$$

$$\psi(\zeta) = K\left[-\frac{4+\gamma}{8} R \log \zeta + \frac{1}{1+2m\zeta} \sum_{r=1}^{3} R'_r \zeta^r\right] \tag{30a}$$

where

$$R = P(0,2), \quad R_r = a_r(0,2), \quad R'_r = C_r(0,2). \tag{30b}$$

Substituting from (29) and (30) in (5a), we obtain the hoop stress

$$\begin{aligned}[\widehat{\theta\theta}]_{\zeta=\sigma} = {} & \frac{\epsilon\omega^2 c^2}{8(1+4m^2+4m\cos\theta)(1-2m^2)}[(\gamma-4)\{n^2(1-2m^2-8m^4)+nm(5-4m^2-16m^4) \\ & + 1+8m^2-18m^4-8m^6\} + 4\{1+6m^2-18m^4-8m^6+2nm(1-4m^2)\} + \\ & + 2\{(\gamma-4)[n(1+2m^2)+3m]+8m\} \times (1-4m^4)\cos\theta + \\ & + 2m\{(\gamma-4)[n(1+2m^2)+2m]+4m\} \times (1-2m^2)\cos 2\theta].\end{aligned} \tag{31}$$

It is noted from (31) that the points of maximum hoop stress are $\theta = 0, \theta = \pi$.

7. Rotation about the normal axis through the centroid

It is easily seen that the x-coordinate of the centroid of the limacon, given by (8), is x_g where

$$x_g = c\left[n + \frac{m}{1+2m^2}\right].$$

The centroid of the limacon will be the point (0,0) when $n = -m(1 + 2m^2)^{-1}$. When the blade is rotating about the axis normal to the plane of the blade and passing through its centroid, the body-force potential is given by

$$V = -\frac{1}{2}\omega^2 z\bar{z} \tag{32}$$

which coincides with (7) for $p = 0, q = 2$. The complex functions for this case are derived to be

$$\phi(\zeta) = K(S_1\zeta + S_2\zeta^2 + S_3\zeta^3 + S_4\zeta^4),$$

$$\psi(\zeta) = K\,\frac{1}{1+2m\zeta}(S'_1\zeta + S'_2\zeta^2 + S'_3\zeta^3) \tag{33a}$$

where

$$S_1 = 3(1-4m^4)^{-1}[(4-\gamma)(1+2m^2)^{-1}(1+4m^2-10m^4-16m^6-8m^8)+4(1+2m^2+2m^4-4m^6)],$$

$$S_2 = 3m(1-4m^4)^{-1}[(4-\gamma)(1+2m^2)^{-1}(1+10m^2-2m^4-24m^6-8m^8)+4(1-10m^4-4m^6)],$$

$$S_3 = 6(\gamma-4)m^2(1-2m^2)(1+2m^2)^{-1}+8m^2,$$

$$S_4 = 6(\gamma-4)m^3(1+2m^2)^{-1},$$

$$S_1' = 12m^2(1+2m^2)^{-2}[(8-\gamma)(1-2m^4)+8m^2]+2m(1+2m^2)^{-1}S_2,$$

$$S_2' = 36m^3(1+2m^2)^{-2}[(4-\gamma)m^2+4(1+3m^2)],$$

$$S_3' = 12(\gamma-8)m^4(1+2m^2)^{-2}. \tag{33b}$$

The hoop stress, for this case, is furnished by

$$[\widehat{\theta\theta}]_{\zeta=\sigma} = \frac{\epsilon\omega^2c^2}{8(1+4m^2+4m\cos\theta)}\left[\frac{1}{(1-4m^4)}\{\gamma(1+2m^2)\times(1+4m^2-10m^4-8m^6) + 16m^4(1-4m^2)\}+4\gamma m(1+2m^2)\cos\theta+2\gamma m^2\cos 2\theta\right]. \tag{34}$$

In this case the points of maximum hoop stress are $\theta = 0, \pi$. The greatest hoop stress occurs at $\theta = \pi$. The minimum hoop stress occurs at $\theta = \theta_m$, given by

$$\cos\theta_m = \frac{1}{4m}[\{1-4m^2)[1+2m^2)^2+64m^4\gamma^{-1}](1-4m^4)^{-1}\}^{1/2}-(1+4m^2)]. \tag{35}$$

Setting $\theta = \pi$ in (34) and simplifying we find the expression for the greatest hoop stress:

$$[\widehat{\theta\theta}]_{\zeta=-1} = \frac{\epsilon\omega^2c^2}{8(1-2m)(1-4m^4)}[\gamma(1-2m+4m^2-2m^4+12m^5-12m^6+8m^7)+ + 16m^4(1+2m)]. \tag{36}$$

The numerical values of the greatest hoop stress for various values of $m \leq 0.45$ ($\eta=\frac{1}{3}$) are given in Table 3 and are illustrated graphically in Fig. 2. The variation of the hoop stress from $\theta = 0$ to $\theta = \pi$ for $m = 0, 0.1, 0.2, 0.3, 0.4$ ($\eta=\frac{1}{3}$) is listed in Table 4 and is shown graphically in Fig. 3.

TABLE 3

Variation of the greatest hoop stress $\widehat{\theta\theta}/\epsilon\omega^2c^2$

m	0	0.1	0.2	0.25	0.3	0.35	0.4	0.45
$[\widehat{\theta\theta}/\epsilon\omega^2c^2]_{\zeta=-1}$	0.166	0.175	0.220	0.278	0.396	0.657	1.326	2.836

TABLE 4

Variation of the hoop stress $\theta\theta/\epsilon\omega^2 c^2$ $(\eta = \frac{1}{2})$

m^{θ}	0°	30°	60°	90°	120°	150°	180°
0	0.166	0.166	0.166	0.166	0.166	0.166	0.166
0.1	0.172	0.171	0.169	0.167	0.168	0.173	0.175
0.2	0.187	0.184	0.176	0.169	0.173	0.197	0.220
0.3	0.209	0.204	0.190	0.176	0.180	0.250	0.396
0.4	0.234	0.226	0.205	0.182	0.177	0.289	1.326

8. Conclusions and discussion

In Sections 3-7 the solution has been obtained for limacon-shaped thin elastic blades rotating with constant angular velocity about the line of symmetry or about any line perpendicular to the line of symmetry and lying in its plane or about a line perpendicular to its plane and passing through the centroid. It has been concluded that the maximum hoop stress occurs at the ends of the line of symmetry in all cases. When the blade is rotating about the line of symmetry or about the normal axis through the centroid the maximum hoop stress increases with m. The comparison of the results of Section 7 with those obtained by Tilley [16] reveals a mistake in his results. It may be noted that the functions $\Psi(\zeta), \lambda'(\zeta)$ of Tilley's paper correspond to $\phi(\zeta)$, $\psi(\zeta)$ of the present paper respectively. The expression for $\lambda'(\zeta)$ given by Equation 27 on page 260 of his paper is

$$\lambda'(\zeta) = \sum_{k=1}^{6} b_k \zeta^k + \frac{\alpha b_7 \zeta^7}{\alpha + \beta\zeta}, \tag{a}$$

the corresponding expression for $\psi(\zeta)$ in our paper is given by (33a). Now (a) and (33a) will be equal only if

$$\alpha b_k + \beta b_{k-1} = 0 \quad (k = 7,6,5,4). \tag{b}$$

The values of b_k given by Equations 34 on pages 263, 264 of his paper do not satisfy the condition (b) above, which shows that the values of b_k calculated by Tilley are wrong. The comparison of the values of S_1', S_2', S_3', given by (33b) with the corresponding values of b_k's given by Equations 34 of Tilley's paper show the simplicity of our results which is due to the reasons explained below. Tilley could have arranged the mapping function, given by Equation (17) of his paper as

$$z = 2\alpha\left(\frac{\beta - h}{2\alpha} + \zeta + \frac{\beta}{2\alpha}\zeta^2\right)$$

or

$$z = 2\alpha(S + \zeta + r\zeta^2), \qquad S = \frac{\beta - h}{2\alpha}, \qquad r = \frac{\beta}{2\alpha}, \tag{c}$$

and since he has taken the centroid of the limacon as the origin of the coordinates, he could have made use of the relation

$$S(1+2r^2)+r=0. \tag{d}$$

By making use of (c) and (d) he could have expressed the values of a_k and b_k in terms of r only, since 2α which corresponds to c of our paper is a dimensional constant and would have remained common. On page 264 of his paper, Tilley has stated that the results have been checked against the known results for a circular rotating disc of radius 2α. This statement is not correct, since by setting $\beta = h = 0$ in Equations 34 of his paper, we find that b_3 does not vanish which shows that $\lambda'(\zeta) \neq 0$ and is disagreeable with the known results.

Part II. CIRCULAR BLADES

We introduce the mapping function

$$z = g(\zeta) = c(n+\zeta), \quad c > 0, \quad 0 \leq n \leq 1, \tag{37}$$

which maps the region occupied by the blade in the z-plane shown in Fig. 4 onto the unit circle in the ζ-plane.

9. Rotation about a chord

When the blade is rotating steadily about the y-axis with angular velocity ω, the body-force potential is the same as given by (26). Setting $m = 0$ in (27), we obtain

$$\phi(\zeta) = K[6(4-\gamma)n \log \zeta + 3\{(4-\gamma)(1+2n^2)+4\}\zeta + (4-\gamma)\zeta^3],$$
$$\psi(\zeta) = -3K[2(4+\gamma)n \log \zeta + (8+3\gamma n^2)\zeta + 4n\zeta^2]. \tag{38}$$

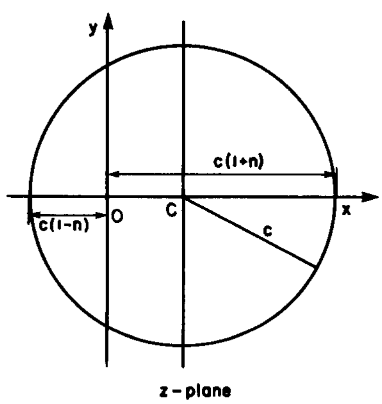

Figure 4. Circular blade.

It is noticed that there will be pressure on the axis of rotation due to non-balance of inertia forces. The resultant $84n$ of the inertia forces will act at $\zeta = 0$, i.e. $x = cn$. Substituting for ϕ and ψ from (38) in (5) we find that

$$\widehat{\rho\rho} = 6K[n^2(\rho^{-2}-1)(4-\gamma) + \frac{1}{2}(1-\rho^2)(8-\gamma)$$
$$+ 2n(\rho^{-1}-\rho)(8-\gamma)\cos\theta + 4(1-\rho^2)\cos 2\theta],$$
$$\widehat{\theta\theta} = 6K[n^2(\rho^{-2}+1)(4-\gamma) - \frac{1}{2}(1-\rho^2)(8-\gamma) - \gamma\rho^2$$
$$+ 2n\{(8-3\gamma)\rho - \gamma\rho^1\}\cos\theta + 4(1-\rho^2)\cos 2\theta],$$
$$\widehat{\rho\theta} = -12K[n\gamma(\rho^{-1}-\rho)\sin\theta + 2(1-\rho^2)\sin 2\theta]. \quad (39)$$

It is easily seen from (39) that $\widehat{\rho\rho} = \widehat{\rho\theta} = 0$ on the boundary i.e. at $\rho = 1$ and the hoop stress is given by

$$[\widehat{\theta\theta}]_{\zeta=\sigma} = \frac{\epsilon\omega^2c^2}{8}[\frac{\gamma}{2} - n(4-\gamma)(n + 2\cos\theta)]. \quad (40)$$

It is seen that the maximum and minimum hoop stresses occur at $\theta = \pi, 0$ respectively and the maximum hoop stress is given by:

$$[\widehat{\theta\theta}]_{\zeta=-1} = \frac{\epsilon\omega^2c^2}{8}[\frac{\gamma}{2} - n(n-2)(4-\gamma)]. \quad (41)$$

10. Rotation about a normal axis

When the blade is rotating uniformly about an axis passing through the origin (i.e. $z = 0$) and normal to its plane the body-force potential is the same as in (29). Putting $m = 0$ in (30), the complex functions are obtained to be

$$\phi(\zeta) = 3K[2(4-\gamma)n\log\zeta + \{4 + (4-\gamma)(1 + n^2)\}\zeta],$$
$$\psi(\zeta) = -6nK[(4+\gamma)\log\zeta + \gamma n\zeta]. \quad (42)$$

Expressions for the stresses for this case are found to be

$$\widehat{\rho\rho} = 6K[(8-\gamma)(1-\rho^2)+n^2(\rho^{-2}-1)(4-\gamma)+2n(\rho^{-1}-\rho)(8-\gamma)\cos\theta],$$
$$\widehat{\theta\theta} = 6K[(8-\gamma)(1-\rho^2)-n^2(\rho^{-2}+1)(4-\gamma)+2\gamma\rho^2 - 2n\{(8-3\gamma)\rho+\gamma\rho^{-1}\}\cos\theta],$$
$$\widehat{\rho\theta} = -12Kn\gamma(\rho^{-1}-\rho)\sin\theta. \quad (43)$$

It is seen from (43) that $\widehat{\rho\rho} = \widehat{\rho\theta} = 0$ at the boundary and that the hoop stress is

$$[\widehat{\theta\theta}]_{\zeta=\gamma} = \frac{\epsilon\omega^2 c^2}{8}\,[\gamma - n(4-\gamma)(n+2\cos\theta)]. \tag{44}$$

The maximum and minimum hoop stresses occur at $\theta = \pi$, 0 respectively, the maximum being given by

$$[\widehat{\theta\theta}]_{\zeta=-1} = \frac{\epsilon\omega^2 c^2}{8}\,[\gamma - n(n-2)(4-\gamma)]. \tag{45}$$

The numerical values of the hoop stress for different positions of the axis of rotation are contained in Table 5 and are illustrated graphically in Figs. 5, 6.

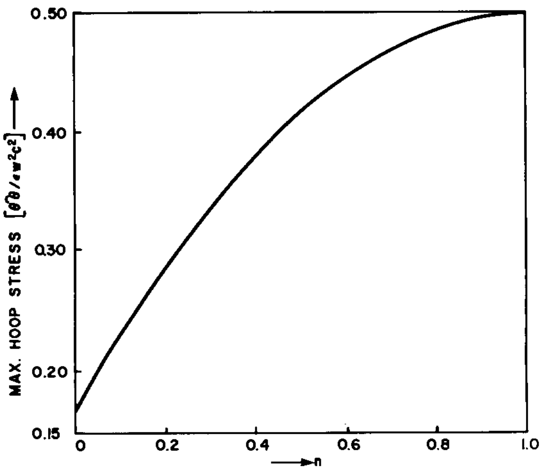

Figure 5. Numerical values of the maximum hoop stress $(\widehat{\theta\theta}/\epsilon\omega^2 c^2)_{\zeta=-1}$ against n for a circular blade rotating about a normal axis passing through the origin 0.

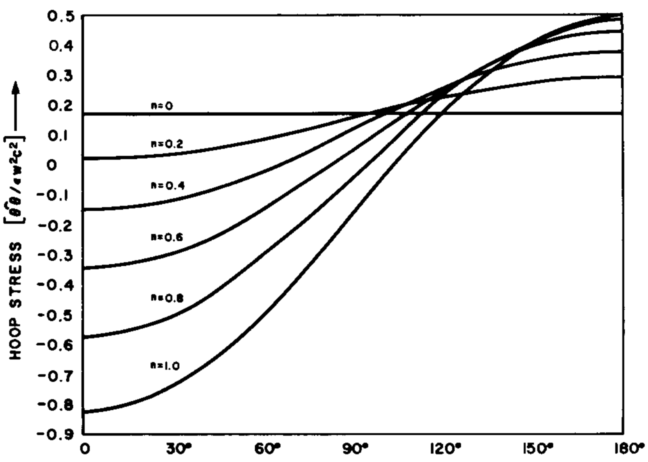

Figure 6. Numerical values of the hoop stress $\widehat{\theta\theta}/\epsilon\omega^2 c^2$ against θ ($0^\circ < \theta < 180^\circ$) for $n = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for a circular blade rotating about a normal axis passing through the origin 0.

TABLE 5

Variation of the hoop stress $\hat{\theta\theta}/\epsilon\omega^2 c^2$ $(\eta = \frac{1}{3})$

n^{θ}	0°	30°	60°	90°	120°	150°	180°
0	0.166	0.166	0.166	0.166	0.166	0.166	0.166
0.1	0.20	0.037	0.086	0.153	0.220	0.269	0.286
0.4	−0.153	−0.118	−0.020	0.113	0.246	0.344	0.380
0.6	−0.353	−0.300	−0.153	0.046	0.346	0.393	0.446
0.8	−0.580	−0.509	−0.313	−0.046	0.220	0.415	0.486
1.0	−0.833	−0.744	−0.500	−0.166	0.166	0.400	0.500

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